View metadata, citation and similar papers at core.ac.uk
brought to you by CORE
provided by CERN Document Server

hep–th/0202039

February 2002

Duality in scalar field theory on noncommutative phase spaces

Edwin Langmann[a] and Richard J. Szabo[b]

[a] *Department of Physics – Mathematical Physics, Royal Institute of Technology*
Stockholm Center for Physics, Astronomy and Biotechnology
S-10691 Stockholm, Sweden
langmann@theophys.kth.se

[b] *Department of Mathematics, Heriot-Watt University*
Riccarton, Edinburgh EH14 4AS, Scotland
R.J.Szabo@ma.hw.ac.uk

Abstract

We describe a novel duality symmetry of Φ^4_{2n}-theory defined on noncommutative Euclidean space and with noncommuting momentum coordinates. This duality acts on the fields by Fourier transformation and scaling. It is an extension, to interactions defined with a star-product, of that which arises in quantum field theories of non-interacting scalar particles coupled to a constant background electromagnetic field. The dual models are in general of the same original form but with transformed coupling parameters, while in certain special cases all parameters are essentially unchanged. We show that this duality persists to all orders of perturbation theory in the full quantum field theory. We also point out various other properties of this class of noncommutative field theories.

1. Introduction. Conventional particle and condensed matter models are often based upon quantum field theories with local interactions. Recently, however, a novel class of non-local field theories has come to the center of attention. These models can be obtained from the standard local ones by defining the contact interactions, which in the standard cases are given by local products of the fields, using the non-local Groenewold-Moyal star-products. The remarkable properties of these models have been under extensive investigation in a variety of different contexts (see [1] for reviews and fairly exhaustive lists of references). One reason for this interest is that these models have a natural interpretation in terms of fields living on a noncommutative spacetime $\mathbb{R}^{2n}$ in which the coordinates $x = (x^1, x^2, \ldots, x^{2n}) = (x^\mu)$ obey the (star-) commutation relations

$$x^\mu \star x^\nu - x^\nu \star x^\mu = -2i\,\theta^{\mu\nu} \ , \tag{1}$$

where $\theta = (\theta^{\mu\nu})$ is a constant $2n \times 2n$ skew-symmetric real matrix. These models also naturally emerge as low-energy effective field theories in string theory with constant background electromagnetic fields [2].

A simple example of such a class of models is provided by noncommutative Φ^4_{2n}-theories, i.e. bosons on $\mathbb{R}^{2n}$ with a quartic interaction in which the local products of the fields are replaced by star-products. The generalization of field theories to ones defined on noncommutative spacetimes naturally suggests the generalization which allows for noncommutative momentum spaces as well. This modification may also be motivated by the observation [2, 3] that such extensions lead, in the case of noncommutative Yang-Mills theory, to actions which are explicitly invariant under Morita duality transformations and they allow one to interpolate between commutative and noncommutative descriptions of the same theory. For scalar field theories it is achieved by coupling the (complex) bosons to an external, constant magnetic field defined by another skew-symmetric $2n \times 2n$ matrix $B = (B_{\mu\nu})$. Similar models have emerged as effective descriptions of some planar condensed matter systems in strong magnetic fields [4], such as quantum Hall models. A similar relativistic scalar field theory has been studied in [5], and its one-particle sector, i.e. the noncommutative Landau problem, in [6]. A further natural generalization allows for an arbitrary flat metric $G = (G_{\mu\nu})$ on $\mathbb{R}^{2n}$ determined by a constant, positive-definite symmetric matrix. In this paper we will show that this class of quantum field theories has a remarkable duality property under Fourier transformation, and we make various other related observations. We will always assume that the matrices B and θ are invertible.

Before stating this duality property of the noncommutative scalar field theory, we recall a well-known toy version of it which occurs in the quantum mechanical harmonic oscillator. Consider the differential operator

$$H = -\frac{\partial^2}{\partial y^2} + \omega^2\, y^2 \equiv H(y;\omega) \ , \tag{2}$$

where $\omega > 0$ and y is a coordinate on the real line. Clearly, under Fourier transformation in y it becomes

$$\hat{H}(p;\omega) = -\omega^2\, \frac{\partial^2}{\partial p^2} + p^2 = -\frac{\partial^2}{\partial (p/\omega)^2} + \omega^2\, (p/\omega)^2 \ , \tag{3}$$

where p is the corresponding momentum space coordinate, or equivalently,

$$\hat{H}(p;\omega) = \omega^2\, H\left(p\,;\,\omega^{-1}\right) = H\left(\omega^{-1}\,p\,;\,\omega\right) \ . \tag{4}$$

This fact can be used to explain the special property of the harmonic oscillator ground state wavefunction that its Fourier transform equals itself, up to a rescaling. This is also the simplest example of a strong-weak coupling duality in a quantum theory.

In a similar manner, we will find that the quantum field theories of charged bosons on noncommutative spacetime and with noncommutative momentum space, parametrized by the matrices G, B and θ as described above, and a coupling constant g, preserve their form under Fourier transformation.[1] More specifically, Fourier transformation and a rescaling

[1] To avoid confusion we point out that, after introducing the magnetic field and the star-product in the action of this model, all field theory computations are done in terms of standard (commutative) position and Fourier variables, and what we mean here is standard Fourier transformation. The noncommutative spaces alluded to above merely provide a useful interpretation of these models.

of the fields amounts to changing the parameters of the field theory as[2]

$$
\begin{array}{rcl}
G & \longmapsto & G \; , \\
B & \longmapsto & B \; , \\
\theta & \longmapsto & -B^{-1}\,\theta^{-1}\,B^{-1} \; , \\
g & \longmapsto & \Big|\det(B\theta)\Big|^{-1/2}\, g \; .
\end{array}
\tag{5}
$$

This property remains true even if one adds a mass term for the scalar fields and allows for both types of quartic star-interaction terms which are possible for charged bosons [7]. Then the mass and the relative weights of these interactions are invariant under this transformation. We interpret Fourier transformation followed by this specific rescaling of the fields (for details see Proposition 1 below) as a 'duality' transformation of the field theory. It maps the model to one of the same form but with some transformed 'coupling parameters' (the parameters characterizing the interaction). Note that the free part of the action, parametrized by the metric, the magnetic field, and the mass, are unchanged by this transformation, and it is consistent to restrict to the case of a trivial metric $G = I$ (the identity matrix). This duality can be regarded as a generalization of the second equality in the harmonic oscillator identity (4). There is also an analog of the first equality which we will discuss in Section 4 (see (41)). It is interesting to note that there are two special points in parameter space at $\theta B = \pm I$ where the coupling parameters are invariant up to a change in sign of θ. For each coupling constant g, the duality maps the field theory with $\theta(B) = +B^{-1}$ to that with $\theta(B) = -B^{-1}$ and thus identifies the two special points. As we will discuss, these 'self-dual' models have remarkable special properties. The significance of the points $\theta B = \pm I$ has been noted in different contexts in [2, 3, 6].

We will prove this duality first at the level of the classical action by straightforward calculation. For the free part, this property is in fact proven by a simple extension of the argument above demonstrating the duality of the harmonic oscillator. The remarkable feature of the duality is that the interaction given by the star-product also possesses this property. We will then argue that this duality carries over to the full quantum field theory. All Green's functions of this model have the property that Fourier transformation is the same as performing the replacements of parameters in (5), replacing the spacetime coordinates x by rescaled Fourier variables $B^{-1}\,k$, and multiplying by $|\det(B)|^{N/2}$, where N is the number of external legs of the given Green's function. In particular, the 'self-dual' models define correlation functions which are invariant under Fourier transformation and rescaling. The crucial point in this extension to the full quantum field theory is the existence of a regularization which is preserved by the duality transformation.

In the next section we give precise definitions of the field theories that we consider in this paper and the proof of the duality at the level of the classical action. In Section 3 we extend the proof to the full quantum field theory. In Section 4 we summarize our results and point out some further interesting properties of these models. Some technical details of the proofs are deferred to two appendices at the end of the paper.

2. The classical action. We consider the field theory of a charged, massive scalar field Φ in Euclidean even-dimensional spacetime $\mathbb{R}^{2n}$ defined by the classical action $S =$

[2] Our conventions for B, θ and g are defined in Section 2.

$S_0 + g^2\, S_{\rm int}$. The free part of the action is given by

$$S_0 = \int d^{2n}x\ \sqrt{\det(G)}\ \Big[\big(G^{-1}\big)^{\mu\nu}\ (P_\mu\Phi)^\dagger(x)(P_\nu\Phi)(x) + m^2\,\Phi^\dagger(x)\Phi(x)\Big]\ , \tag{6}$$

where m is the mass parameter, and

$$(P_\mu\Phi)(x) \equiv (-i\,\partial_\mu - B_{\mu\nu}\,x^\nu)\,\Phi(x) \tag{7}$$

with $\partial_\mu = \partial/\partial x^\mu$. The interaction part is

$$S_{\rm int} = \int d^{2n}x\ \sqrt{\det(G)}\ \Big[\alpha\,(\Phi^\dagger\star\Phi\star\Phi^\dagger\star\Phi)(x) + \beta\,(\Phi^\dagger\star\Phi^\dagger\star\Phi\star\Phi)(x)\Big]\ , \tag{8}$$

where

$$(f_1\star f_2)(x) = f_1(x)\ \exp\Big(-i\,\overleftarrow{\partial}_\mu\,\theta^{\mu\nu}\,\overrightarrow{\partial}_\nu\Big)\ f_2(x) \tag{9}$$

is the Groenewold-Moyal star-product. Note that we differ by factors of 2 from the usual conventions for B and θ, in order to simplify some of the formulas which follow. The action S_0 describes scalar bosons of mass m coupled to a constant external electromagnetic field $F_{\mu\nu} = 2B_{\mu\nu}$. Since $[P_\mu, P_\nu] = -2i\,B_{\mu\nu}$, one can also interpret B as a parameter which produces noncommuting momentum space coordinates. In the action $S_{\rm int}$ we have included the two inequivalent, noncommutative quartic interactions of a complex scalar field [7] which we weight by the real parameters α and β.

We will now give a precise formulation of the duality in the classical field theory.

Proposition 1 (Classical duality): *The action*[3]

$$S = S_0 + g^2\,S_{\rm int} \equiv S[\Phi; B, g, \theta] \tag{10}$$

defined above obeys

$$S[\Phi; B, g, \theta] = S\Big[\tilde\Phi\,;\,B\,,\,\tilde g\,,\tilde\theta\Big]\ , \tag{11}$$

where

$$\tilde\Phi(x) = \sqrt{\big|\det(B)\big|}\ \hat\Phi(B\,x) \tag{12}$$

and

$$\hat\Phi(k) = (2\pi)^{-n}\int_{\mathbb{R}^{2n}} d^{2n}x\ \ {\rm e}^{-ik\cdot x}\,\Phi(x)\ ,\quad k\cdot x = k_\mu x^\mu \tag{13}$$

is the Fourier transform of $\Phi(x)$. The transformed coupling parameters are

$$\tilde\theta = -B^{-1}\,\theta^{-1}\,B^{-1}\ ,\quad \tilde g = \big|\det(B\theta)\big|^{-1/2}\,g\ . \tag{14}$$

Moreover, the transformation $(\Phi; B, g, \theta) \mapsto (\tilde\Phi; B, \tilde g, \tilde\theta)$ is a duality of the field theory, i.e. it generates a cyclic group of order two.

[3] We assume here that Φ is a Schwartz test function on $\mathbb{R}^{2n}$ for simplicity. In the notation we explicitly indicate only the parameters B, g and θ which are affected by or are involved in the duality transformations and suppress the dependence on G, α, β and m which remain unchanged. The notation $S[\Phi; \ldots]$ is shorthand for $S[\Phi, \Phi^\dagger; \ldots]$.

We will first establish the duality symmetry of S_0 with $m = 0$. We use the Parseval relation to rewrite S_0 in momentum space, i.e. in terms of the Fourier transform $\hat{\Phi}$ of Φ. We have

$$(\widehat{P_\mu \Phi})(k) = \left(k_\mu - i\, B_{\mu\nu}\, \hat{\partial}^\nu\right) \hat{\Phi}(k) = \left(i\, \tilde{\partial}_\mu + B_{\mu\nu}\, \tilde{k}^\nu\right) \hat{\Phi}(k) \ , \tag{15}$$

where $\hat{\partial}^\mu = \partial/\partial k_\mu$, $\tilde{k}^\mu = (B^{-1})^{\mu\nu}\, k_\nu$, and $\tilde{\partial}_\mu = \partial/\partial \tilde{k}^\mu$. Thus the change of variables $k \mapsto \tilde{k}$ yields

$$S_0 = \left|\det(B)\right| \int d^{2n}\tilde{k}\ \sqrt{\det(G)}\ \left(G^{-1}\right)^{\mu\nu}\ \left(\tilde{P}_\mu \hat{\Phi}\right)^\dagger \left(B\, \tilde{k}\right) \left(\tilde{P}_\nu \hat{\Phi}\right) \left(B\, \tilde{k}\right) \ , \tag{16}$$

where

$$\tilde{P}_\mu = -i\, \tilde{\partial}_\mu - B_{\mu\nu}\, \tilde{k}^\nu \ . \tag{17}$$

Changing the name of the integration variable to $\tilde{k} = x$, this action has same form as S_0 in (6) except that $\Phi(x)$ is replaced by the field $\tilde{\Phi}(x)$ defined in (12). This proves that $S_0[\Phi; B] = S_0[\tilde{\Phi}; B]$ for $m = 0$. The proof of invariance of the mass term

$$m^2 \int d^{2n}x\ \sqrt{\det(G)}\, \Phi^\dagger(x)\Phi(x) \equiv m^2\, \sqrt{\det(G)}\ \langle \Phi, \Phi \rangle \tag{18}$$

is similar but simpler. Using the Parseval relation and performing the transformation $k \mapsto x = B^{-1}\, k$ as above one finds $\langle \Phi, \Phi \rangle\ =\ \langle \tilde{\Phi}, \tilde{\Phi} \rangle$, as claimed.

We now turn to the proof the duality property of the action $S_{\rm int}$. The crucial step is the behaviour of $S_{\rm int}$ under Fourier transformation of the fields,

$$S_{\rm int}[\Phi; G, \theta] = \left|\det(\theta)\right|^{-1} S_{\rm int}\left[\hat{\Phi}\, ;\, G\, ,\, \theta^{-1}\right] \ , \tag{19}$$

i.e. Fourier transformation yields an action of the same form (8) but with the Groenewold-Moyal product in momentum space defined as

$$\left(\hat{f}_1 \,\hat{\star}\, \hat{f}_2\right)(k) = \hat{f}_1(k)\ \exp\left(-i\, \overleftarrow{\hat{\partial}^\mu}\, \left(\theta^{-1}\right)_{\mu\nu}\, \overrightarrow{\hat{\partial}^\nu}\right)\, \hat{f}_2(k) \ . \tag{20}$$

We first note that this readily implies the duality properties stated in section 1. Inserting $\hat{\Phi}(k) = |\det(B)|^{-1/2}\, \tilde{\Phi}(B^{-1}\, k)$ and changing variables to $x = B^{-1}\, k$ gives

$$S_{\rm int}\left[\hat{\Phi}\, ;\, G\, ,\, \theta^{-1}\right] = \left|\det(B)\right|^{-1} S_{\rm int}\left[\tilde{\Phi}\, ;\, G\, ,\, -B^{-1}\, \theta^{-1}\, B^{-1}\right] \ , \tag{21}$$

where we have inserted $\hat{\partial}^\mu = -(B^{-1})^{\mu\nu}\, \partial_\nu = \partial_\nu\, (B^{-1})^{\nu\mu}$ into (20). This yields

$$S_{\rm int}[\Phi; \theta] = \left|\det(\theta B)\right|^{-1} S_{\rm int}\left[\tilde{\Phi}\, ;\, -B^{-1}\, \theta^{-1}\, B^{-1}\right] \ , \tag{22}$$

implying the third and fourth transformation rules in (5).

We now prove (19). For this, we write $S_{\rm int} = \sqrt{\det(G)}\, (\alpha\, S_\alpha + \beta\, S_\beta)$ and first treat the action S_α. This interaction term has a simple form in momentum space,

$$S_\alpha = \prod_{a=1}^4 \int \frac{d^{2n}k_a}{(2\pi)^n}\ \hat{\Phi}^\dagger(k_1)\hat{\Phi}(k_2)\hat{\Phi}^\dagger(k_3)\hat{\Phi}(k_4)\, \hat{V}(k_1, k_2, k_3, k_4) \ , \tag{23}$$

with vertex function

$$\hat{V}(k_1,k_2,k_3,k_4) = \int d^{2n}x \, \left(\mathrm{e}^{-ik_1\cdot x} \star \mathrm{e}^{ik_2\cdot x}\right)\left(\mathrm{e}^{-ik_3\cdot x} \star \mathrm{e}^{ik_4\cdot x}\right)$$
$$= (2\pi)^{2n}\,\delta^{2n}(k_1-k_2+k_3-k_4)\;\mathrm{e}^{-i\theta^{\mu\nu}\left[(k_1)_\mu(k_2)_\nu+(k_3)_\mu(k_4)_\nu\right]} \,. \tag{24}$$

In a similar manner we can write the interaction term in position space,

$$S_\alpha = \prod_{a=1}^{4}\int \frac{d^{2n}x_a}{(2\pi)^n}\;\Phi^\dagger(x_1)\Phi(x_2)\Phi^\dagger(x_3)\Phi(x_4)\,V(x_1,x_2,x_3,x_4)\;, \tag{25}$$

and compute the position space vertex function by inverse Fourier transformation,

$$V(x_1,x_2,x_3,x_4) = \prod_{a=1}^{4}\int \frac{d^{2n}k_a}{(2\pi)^n}\;\mathrm{e}^{\,i(k_1\cdot x_1-k_2\cdot x_2+k_3\cdot x_3-k_4\cdot x_4)}\,\hat{V}(k_1,k_2,k_3,k_4)\;. \tag{26}$$

To appreciate the result of this integration, it is useful to recall that in the conventional case ($\theta = 0$) the interaction vertices in position and momentum space are very different. The former one is fully local, $V(x_1,x_2,x_3,x_4) \propto \delta^{2n}(x_1-x_2+x_3-x_4)\,\delta^{2n}(x_1-x_2)\,\delta^{2n}(x_2-x_3)$, while the latter one is non-local, $\hat{V}(k_1,k_2,k_3,k_4) \propto \delta^{2n}(k_1-k_2+k_3-k_4)$. However, for non-singular θ, both vertex functions have the same non-local form. Using the representation $(2\pi)^{2n}\,\delta^{2n}(k_1-k_2+k_3-k_4) = \int d^{2n}t\;\mathrm{e}^{-it\cdot(k_1-k_2+k_3-k_4)}$, the resulting momentum integrals in (26) are all Gaussian, and a straightforward computation yields (for details see Appendix A)

$$V(x_1,x_2,x_3,x_4) = \frac{(2\pi)^{2n}}{\left|\det(\theta)\right|}\,\delta^{2n}(x_1-x_2+x_3-x_4)\;\mathrm{e}^{-i(\theta^{-1})_{\mu\nu}\left[(x_1)^\mu(x_2)^\nu+(x_3)^\mu(x_4)^\nu\right]}\;, \tag{27}$$

which, up to the factor $|\det(\theta)|^{-1}$, has the same form as the momentum space vertex function (24) but with the matrix θ replaced by its inverse. This implies that $S_\alpha[\Phi;\theta] = |\det(\theta)|^{-1}\,S_\alpha[\hat{\Phi};\theta^{-1}]$.

To see that the same property is true of the interaction term S_β, we note that it can be written as

$$S_\beta = \prod_{a=1}^{4}\int \frac{d^{2n}k_a}{(2\pi)^n}\;\hat{\Phi}^\dagger(k_1)\hat{\Phi}^\dagger(k_2)\hat{\Phi}(k_3)\hat{\Phi}(k_4)\,\hat{V}(k_1,-k_2,-k_3,k_4)$$
$$= \prod_{a=1}^{4}\int \frac{d^{2n}x_a}{(2\pi)^n}\;\Phi^\dagger(x_1)\Phi^\dagger(x_2)\Phi(x_3)\Phi(x_4)\,V(x_1,-x_2,-x_3,x_4)\;, \tag{28}$$

with the *same* vertex functions V and $\hat{V}$ as above. The sign changes allowing us to use the results in (24), (26) and (27) all work out since $k_1\cdot x_1 + k_2\cdot(-x_2) - k_3\cdot(-x_3) - k_4\cdot x_4 = k_1\cdot x_1 - k_2\cdot x_2 + k_3\cdot x_3 - k_4\cdot x_4$. This proves (19). It is also evident that the transformation defined in Proposition 1 is equal to its inverse, which thereby completes the proof of Proposition 1.

3. Quantization. We will now show that this duality property holds true at the full quantum level as well. Formally, the quantum field theory is defined by the functional integral

$$Z[J] = \int \mathcal{D}\Phi\;\mathcal{D}\Phi^\dagger\;\mathrm{e}^{-S[\Phi;B,g,\theta]+\langle\Phi,J\rangle+\langle J,\Phi\rangle}\;, \tag{29}$$

where $\langle \Phi, J\rangle = \int d^{2n}x\ \Phi^\dagger(x)\, J(x)$ and similarly for $\langle J,\Phi\rangle$. Of course, a proper definition requires a specification of ultraviolet and infrared regularizations, which will be described below. Here the external source fields $J(x)$ and $J^\dagger(x)$ are regarded as independent functions. The generating functional of all connected Green's functions is then given by

$$\mathcal{G}(J) = -\ln\frac{Z[J]}{Z[0]} \equiv \mathcal{G}(J;B,g,\theta)\ . \tag{30}$$

The path integral measure $\mathcal{D}\Phi\ \mathcal{D}\Phi^\dagger$ is defined so that for $g=0$, $\mathcal{G}(J)=\langle J, CJ\rangle$, where C is the propagator of the quantum field theory which is given in position space by the free two-point correlation function

$$C(x,y) = \langle x|\left(P^2+m^2\right)^{-1}|y\rangle\ . \tag{31}$$

Since the functional integration measure is invariant under the transformation $\Phi\mapsto\tilde\Phi$, and

$$\langle\Phi,J\rangle = \int d^{2n}k\ \hat\Phi^\dagger(k)\,\hat J(k) = \left|\det(B)\right|^{-1}\int d^{2n}k\ \tilde\Phi^\dagger\left(B^{-1}\,k\right)\,\tilde J\left(B^{-1}\,k\right) = \left\langle\tilde\Phi\,,\,\tilde J\right\rangle \tag{32}$$

along with the analogous property for $\langle J,\Phi\rangle$, the change of variables $\Phi\mapsto\tilde\Phi$ in the path integral and the duality property of the action (Proposition 1) formally yield the identity

$$\mathcal{G}(J;B,g,\theta) = \mathcal{G}\left(\tilde J\,;\,B\,,\,\tilde g\,,\,\tilde\theta\right)\ . \tag{33}$$

This implies that any connected Green's function with N external legs,

$$\mathcal{G}_{n,N-n}(x_1,\dots,x_N) = \prod_{c=1}^n\frac{\delta}{\delta J(x_c)}\ \prod_{c=n+1}^N\frac{\delta}{\delta J^\dagger(x_c)}\,\mathcal{G}(J)\Bigg|_{J=J^\dagger=0}\ , \tag{34}$$

obeys the identity

$$\hat{\mathcal{G}}_{n,N-n}(k_1,\dots,k_N;B,g,\theta) = \left|\det(B)\right|^{N/2}\,\mathcal{G}_{n,N-n}\left(B^{-1}\,k_1\,,\,\dots\,,\,B^{-1}\,k_N\,;\,B\,,\,\tilde g\,,\,\tilde\theta\right) \tag{35}$$

where $\hat{\mathcal{G}}_{n,N-n}$ is the Fourier transform of $\mathcal{G}_{n,N-n}$,

$$\hat{\mathcal{G}}_{n,N-n}(k_1,\dots,k_N) = \prod_{c=1}^n\int\frac{d^{2n}x_c}{(2\pi)^n}\ \mathrm{e}^{\,-ik_c\cdot x_c}\ \prod_{c=n+1}^N\int\frac{d^{2n}x_c}{(2\pi)^n}\ \mathrm{e}^{\,ik_c\cdot x_c}\,\mathcal{G}_{n,N-n}(x_1,\dots,x_N)\ . \tag{36}$$

The relation (35) is particularly interesting for $B\theta=\pm\,I$, since then $\tilde g=g$ and $\tilde\theta=-\theta$. In this case, Fourier transformation leaves all correlation functions invariant up to rescaling and sign change of θ.

To substantiate this argument we now show that there is a natural regularization of the quantum field theory which cures all possible divergences and which is also invariant under the duality transformation. With this regularization included, all formal manipulations above are put on solid ground, and they prove the stated conclusions for the regulated Green's functions. The duality invariant regularization amounts to replacing the propagator (31) by the following regulated one

$$C_\Lambda(x,y) = \langle x|\left(P^2+m^2\right)^{-1}\ \mathrm{f}\left(\Lambda^{-2}\left[P^2+Q^2\right]\right)|y\rangle\ , \tag{37}$$

where

$$\frac{1}{2}\left(P^2+Q^2\right) \equiv \frac{1}{2}\left(G^{-1}\right)^{\mu\nu}\left(P_\mu P_\nu + Q_\mu Q_\nu\right) = -\partial_\mu\,\partial^\mu + (Bx)_\mu\,(Bx)^\mu \tag{38}$$

with $P_\mu = -i\,\partial_\mu - B_{\mu\nu}\,x^\nu$ (as in (7)) and $Q_\mu = -i\,\partial_\mu + B_{\mu\nu}\,x^\nu$. The cut-off function $\mathrm{f}(s)$ is defined for real positive s such that $\mathrm{f}(0) = 1$ (so that $C_\Lambda \to C$ as $\Lambda \to \infty$) and such that it decays sufficiently fast as $s \to \infty$. For definiteness, we will assume that $\mathrm{f}(s)$ is smooth, monotonically decreasing, and such that $\mathrm{f}(s) = 1$ for $0 \le s \le 1$ and $\mathrm{f}(s) = 0$ for $s \ge 2$. Heuristically, we would expect that this modification is enough to regulate all potential ultraviolet and infrared divergences of the theory. The term $-\partial_\mu\,\partial^\mu$ in the argument of f cuts off the high momentum modes, while the term $(Bx)_\mu\,(Bx)^\mu$ should regulate all long-distance divergences. In Appendix B we present the formal arguments which substantiate this reasoning. Here we only show that this regularization is indeed invariant under the duality transformation.

In Section 2 we showed that, under Fourier transformation, the differential operator P_μ becomes $\hat{P}_\mu = k_\mu - i\,B_{\mu\nu}\,\hat{\partial}^\nu$, and under the change of variables $k \mapsto x = B^{-1}\,k$ this transforms back to $-P_\mu$. Similarly, the duality transformation maps $Q_\mu \mapsto -Q_\mu$. Thus

$$\mathrm{f}\left(\Lambda^{-2}\left[P^2+Q^2\right]\right) \longmapsto \mathrm{f}\left(\Lambda^{-2}\left[P^2+Q^2\right]\right) \tag{39}$$

under the duality for arbitrary cut-off functions f. This proves invariance of the regularization procedure.

The results of this section can be summarized as follows.

Proposition 2 (Quantum duality): *The regularization defined above is invariant under the duality transformation given in Proposition 1. Moreover, with this regularization and for Euclidean spacetime metric $G = I$, all Feynman diagrams of the quantum field theory defined by (10) can be represented as* finite *sums and are thus convergent. The corresponding regulated generating functional $\mathcal{G}_\Lambda$ of all connected Green's functions is therefore well-defined, and it possesses the duality symmetry*

$$\mathcal{G}_\Lambda(J;B,g,\theta) = \mathcal{G}_\Lambda\left(\tilde{J}\,;\,B\,,\,\tilde{g}\,,\,\tilde{\theta}\right)\ , \tag{40}$$

where $\tilde{J}(x) = \sqrt{|\det(B)|}\ \hat{J}(B\,x)$ and $\hat{J}$ is the Fourier transform of J.

4. Summary and discussion. We have considered a class of interacting scalar quantum field theories in which the positions and momenta are noncommuting coordinates. We showed that these models are invariant under Fourier transformation, rescaling, and some specific changes of the coupling parameters. These transformations generate the action of a group of order two on the coupling parameter space and thereby define a duality of the quantum field theory. This duality property was first shown on the level of classical actions (Proposition 1) and then extended to the full quantum level (Proposition 2). The quantum result was proven in perturbation theory, for the special case of a trivial metric $G = I$, and only in the regulated quantum field theory (with finite cut-offs). We conjecture that this result extends to the cases with more general Euclidean metrics G on $\mathbb{R}^{2n}$ and also beyond perturbation theory. Along the way we have also made use of an interesting technical aspect of these models, the fact that it is convenient to perform the corresponding quantum field theoretic computations in 'Landau space' (rather than in position or momentum space), i.e. by expanding in the basis of Landau wavefunctions $\phi_{\mathbf{n}}$ as described in Appendix B. This results in somewhat unusual expressions for the Feynman diagrams. Similar techniques have also been applied in [5, 6].

It would be interesting to study in detail what happens to this duality property as the cut-off $\Lambda \to \infty$ is removed. The results presented here can shed interesting light on the nature of divergences in noncommutative quantum field theory, such as UV/IR mixing [8]. In particular, the duality symmetry shows explicitly the well-known result that the correlation functions of the model do not have a continuous $\theta \to 0$ limit. As discussed in section 3, the regularization parameter Λ can be regarded at the same time as both an ultraviolet and an infrared cut-off. However, one can generalize the regularization procedure by changing the argument of the cut-off function f in (37) from $\Lambda^{-2}\,[-\partial^2+(B\,x)^2]$ to $-\Lambda_{\rm UV}^{-2}\,\partial^2+\Lambda_{\rm IR}^{-2}\,(B\,x)^2$, i.e. use different parameters $\Lambda_{\rm UV}$ and $\Lambda_{\rm IR}$ to cut-off high momenta and long wavelengths, respectively. In conventional quantum field theories one would interpret divergences which arise in the limit $\Lambda_{\rm UV} \to \infty$ as ultraviolet divergences and those for $\Lambda_{\rm IR} \to \infty$ as infrared divergences. However, in the present noncommutative quantum field theory such a distinction between divergences appears to be somewhat artificial. The duality transformation exchanges the parameters $\Lambda_{\rm UV}$ and $\Lambda_{\rm IR}$ but otherwise leads to a model of the same kind, i.e. ultraviolet divergences turn into infrared divergences and vice versa. We expect that a more physical interpretation of the divergences arises if one works consistently in position space. We believe that this duality effect should also have some bearing on the issue of renormalizability of this class of quantum field theories.

We conclude by pointing out a few further interesting aspects of these models:

- There is in fact a much larger group of symmetry transformations associated with these noncommutative field theories. All linear coordinate transformations $x \mapsto L\,x$, with L an invertible $2n \times 2n$ real-valued matrix, map the field theory to one of the same kind but with altered coupling parameters. In particular, the class of models with $G = I$ are left invariant by orthogonal transformations $L \in O(2n, \mathbb{R})$, but of course both θ and B change under these transformations.

- It is easy to see that Fourier transformation alone, without rescaling the arguments of the fields, is also a duality transformation of the present class of models. It acts on the parameters of the field theory as

$$\begin{aligned} B &\longmapsto B^{-1}\ , \\ G &\longmapsto -B^{-1}\,G\,B^{-1}\ , \\ \theta &\longmapsto \theta^{-1}\ , \\ g &\longmapsto \Bigl|\det(B\theta)\Bigr|^{-1/2}\,g\ , \end{aligned} \tag{41}$$

and now also affects the free part of the action. This duality can be regarded as a generalization of the first equality in the harmonic oscillator identity (4). Note that in this transformation, the roles of covariant and contravariant tensors on $\mathbb{R}^{2n}$ are interchanged, in contrast to the previous duality transformation. This is a simple consequence of Fourier transformation which maps $x^\mu \mapsto k_\mu$, and it is reminescent of what occurs in standard T-duality transformations. In fact, at the special points where $B = \pm\,\theta^{-1}$, the transformations (41) are precisely those obtained from the zero rank limit of the usual gauge Morita duality relations between noncommutative Yang-Mills theories on $2n$-dimensional tori [1]–[3]. This remarkable coincidence can be understood by interpreting the noncommutative Φ^4_{2n}-theory as a discrete $\mathbb{Z}_2$ noncommutative gauge theory [9], and it is another sign of the inherently stringy nature of noncommutative quantum field theories [1].

- In the special cases $\theta B = \pm\,I$ and $G = I$ the interaction vertices $v(\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3, \mathbf{n}_4)$ defined in Appendix B simplify considerably, and the resulting model has a natural interpretation as a matrix model. Based on preliminary results [9] we conjecture that these models are exactly solvable in $2n = 2$ dimensions. Moreover, the noncommutative soliton equations derived from this model, including the free part of the action, at $\theta B = \pm\,I$ are straightforward to solve, and the resulting soliton profiles are the same as those which arise when the kinetic energy term is neglected [10].

Acknowledgments

We thank C.-S. Chu, J. Gracia-Bondía, G. Landi, U. Lindström, F. Lizzi, M. Salmhofer and K. Zarembo for helpful discussions and correspondence. This work was supported in part by the Swedish Science Research Council (VR) and the Göran Gustafssons Foundation. The work of R.J.S. was supported in part by an Advanced Fellowship from the Particle Physics and Astronomy Research Council (U.K.).

Appendix A. To compute the vertex function V defined by (26) and (24), it is convenient to introduce variables in $\mathbb{R}^{8n}$ defined by

$$K = (k_1, -k_2, k_3, -k_4) \ , \quad X = (x_1, x_2, x_3, x_4) \ . \tag{42}$$

They allow us to write $V(X) = (2\pi)^{-4n} \int d^{8n}K \ \hat V(K) \ \mathrm{e}^{\,-iK\cdot X}$, where

$$\hat V(K) = (2\pi)^{2n} \, \delta^{2n}(k_1 - k_2 + k_3 - k_4) \ \mathrm{e}^{\,-\frac12\, K\cdot \mathsf{A}_\theta K} \tag{43}$$

and A_θ is the skew-symmetric $8n \times 8n$ matrix

$$\mathsf{A}_\theta = \begin{pmatrix} 0 & i\,\theta & 0 & 0 \\ -i\,\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & i\,\theta \\ 0 & 0 & -i\,\theta & 0 \end{pmatrix} \ . \tag{44}$$

Inserting $(2\pi)^{2n}\,\delta^{2n}(k_1 - k_2 + k_3 - k_4) = \int d^{2n}t \ \mathrm{e}^{\,-iK\cdot T}$ with $T = (t,t,t,t) \in \mathbb{R}^{8n}$ and interchanging the order of integrations gives the Gaussian integral

$$V(X) = \int d^{2n}t \int \frac{d^{8n}K}{(2\pi)^{4n}} \ \mathrm{e}^{\,-iK\cdot(T+X) - \frac12\, K\cdot \mathsf{A}_\theta K} = \det(\theta)^{-2} \int d^{2n}t \ \mathrm{e}^{\,\frac12\,(T+X)\cdot \mathsf{A}_{\theta^{-1}}(T+X)} \ , \tag{45}$$

where we have used $(\mathsf{A}_\theta)^{-1} = \mathsf{A}_{\theta^{-1}}$ and $\det(\mathsf{A}_\theta) = \det(\theta)^4$. Now $T \cdot \mathsf{A}_{\theta^{-1}} T = 0$ and $X \cdot \mathsf{A}_{\theta^{-1}} T = T \cdot \mathsf{A}_{\theta^{-1}} X = -i\,(x_1 - x_2 + x_3 - x_4) \cdot \theta^{-1}\,t$, and so the t-integral in (45) yields

$$V(X) = \det(\theta)^{-1}\,(2\pi)^{2n}\,\delta^{2n}(x_1 - x_2 + x_3 - x_4)\ \mathrm{e}^{\frac{1}{2}X\cdot\mathsf{A}_{\theta^{-1}}X}\ , \tag{46}$$

where one factor of $\det(\theta)^{-1}$ has been eliminated by the change of variables $t \mapsto \theta^{-1}\,t$. From (43), and the fact that the change of sign in the exponential in (46) is exactly compensated by the different definitions of the vectors K and X in (42), it follows that the expression (46) is identical to (27).

Appendix B. In this appendix we will prove that all Feynman diagrams computed in the regulated quantum field theory, as defined in Section 3, are given by finite sums. For simplicity we assume $G = I$. We believe that the extension of the argument to general constant Euclidean metrics G is straightforward.

In the coordinate system where B has the Jordan normal form

$$(B_{\mu\nu}) = \begin{pmatrix} 0 & B_1 & & & \\ -B_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & B_n \\ & & & -B_n & 0 \end{pmatrix}\ , \qquad B_j > 0 \tag{47}$$

we have

$$P^2 \equiv P_\mu P^\mu = H_{12} + H_{34} + \ldots + H_{2n-1,2n}\ , \tag{48}$$

where, for each $j = 1,\ldots,n$, $H_{2j-1,2j}$ depends only on the spacetime coordinates x_{2j-1} and x_{2j}. It is identical to the Landau Hamiltonian, i.e. the quantum mechanical Hamiltonian describing the motion of a particle in the two-dimensional (x_{2j-1}, x_{2j})-plane with an applied, constant perpendicular magnetic field B_j. This Hamiltonian has well-known eigenfunctions ϕ_{m_j,ℓ_j} which are labelled by non-negative integers m_j and ℓ_j, and obey $H_{2j-1,2j}\,\phi_{m_j,\ell_j} = B_j(m_j + \frac{1}{2})\,\phi_{m_j,\ell_j}$. The eigenfunctions of the operator $P_\mu P^\mu$ are therefore given by

$$\phi_{\mathbf{n}}(x) = \prod_{j=1}^{n} \phi_{m_j,\ell_j}(x_{2j-1}, x_{2j})\ , \quad \mathbf{n} \equiv (\mathbf{m}, \mathbf{l}) = (m_1,\ldots,m_n,\ell_1,\ldots,\ell_n)\ , \tag{49}$$

with $P^2\,\phi_{\mathbf{n}} = E(\mathbf{n})\,\phi_{\mathbf{n}}$. One can check that they are also eigenfunctions of the operator Q^2, $Q^2\,\phi_{\mathbf{n}} = F(\mathbf{n})\,\phi_{\mathbf{n}}$. The eigenvalues are given by

$$E(\mathbf{n}) = \sum_{j=1}^{n} B_j\left(m_j + \frac{1}{2}\right)\ , \quad F(\mathbf{n}) = \sum_{j=1}^{n} B_j\left(\ell_j + \frac{1}{2}\right)\ , \tag{50}$$

and they each depend on only half of the quantum numbers $\mathbf{n} = (\mathbf{m}, \mathbf{l})$. This is just a generalization of the well-known degeneracy of the energy eigenstates of the Landau problem. Note that complex conjugation of $\phi_{\mathbf{n}}$ amounts to interchanging P^2 and Q^2.

This basis allows one to diagonalize the free part of the action and is therefore convenient for computing all Green's functions of the model, i.e. all fields can be labelled by a

$2n$-vector of non-negative integers $\mathbf{n}$. By expanding the scalar fields in this orthonormal basis,

$$\Phi(x) = \sum_{\mathbf{n}} \phi^\dagger_{\mathbf{n}}(x)\, A(\mathbf{n}) \ , \qquad \Phi^\dagger(x) = \sum_{\mathbf{n}} \phi_{\mathbf{n}}(x)\, A^\dagger(\mathbf{n}) \ , \tag{51}$$

the free part of the action can be written as

$$S_0 = \sum_{\mathbf{n}} \Big[E(\mathbf{n}) + m^2\Big]\, A^\dagger(\mathbf{n}) A(\mathbf{n}) \ . \tag{52}$$

The expansion of the interaction part of the action gives

$$S_{\rm int} = \sum_{\mathbf{n}_1,\mathbf{n}_2,\mathbf{n}_3,\mathbf{n}_4} v(\mathbf{n}_1,\mathbf{n}_2,\mathbf{n}_3,\mathbf{n}_4)\, A^\dagger(\mathbf{n}_1)A(\mathbf{n}_2)A^\dagger(\mathbf{n}_3)A(\mathbf{n}_4) \ , \tag{53}$$

where the interaction vertices $v(\mathbf{n}_1,\mathbf{n}_2,\mathbf{n}_3,\mathbf{n}_4)$ can be evaluated by straightforward calculation from the position space vertex functions $V(x_1,x_2,x_3,x_4)$ given in Section 2. The only property of these vertices that we shall need for the present argument is that they are all well-defined.

From (53) we see that the propagator in this representation is diagonal and given by $c(\mathbf{n}) = [E(\mathbf{n}) + m^2]^{-1}$. Moreover, the duality-symmetric regularization amounts to replacing it by

$$c_\Lambda(\mathbf{n}) = \frac{1}{E(\mathbf{n}) + m^2}\ \mathrm{f}\left(\Lambda^{-2}\Big[E(\mathbf{n}) + F(\mathbf{n})\Big]\right) \ . \tag{54}$$

In this representation, all Feynman diagrams of the quantum field theory are of the schematic form $\sum_{\mathbf{n}'_1,\dots,\mathbf{n}'_K} \prod_k c_\Lambda(\mathbf{n}'_k)(\cdots)$, with $(\cdots)$ a product of interaction vertices v, which depend on the summation variables $\mathbf{n}'_k$, and propagators c, which depend on the labels $\mathbf{n}_k$ of external legs that are not summed over. With the assumed properties of the cut-off function f stated in Section 3, the propagator $c_\Lambda(\mathbf{n})$ is nonzero only if $E(\mathbf{n})+F(\mathbf{n}) = \sum_j B_j(m_j+\ell_j+1) < 2\Lambda^2$, which at finite Λ is true for only a finite number of distinct $\mathbf{n}$'s. Thus all Feynman diagrams are represented by finite sums. Note that these sums give the Green's functions $\mathcal{G}_{n,N-n}(\mathbf{n}_1,\dots,\mathbf{n}_N)$ in the basis $\phi_{\mathbf{n}}$. From $\mathcal{G}_{n,N-n}(\mathbf{n}_1,\dots,\mathbf{n}_N)$ one can compute the corresponding position space Green's function by multiplying it with the functions $\phi_{\mathbf{n}_k}(x_k)$, $k = 1,\dots,n$, and $\phi^\dagger_{\mathbf{n}_k}(x_k)$, $k = n+1,\dots,N$, and summing over all $\mathbf{n}_k$. As above, one finds that these sums are all finite and are thus well-defined, which completes the argument.

References

[1] A. Konechny and A. Schwarz, [hep-th/0012145; 0107251]; M.R. Douglas and N.A. Nekrasov, Rev. Mod. Phys. 73 (2002) 977 [hep-th/0106048]; R.J. Szabo, [hep-th/0109162].

[2] N. Seiberg and E. Witten, J. High Energy Phys. 9909 (1999) 032 [hep-th/9908142].

[3] B. Pioline and A. Schwarz, J. High Energy Phys. 9908 (1999) 021 [hep-th/9908019]; J. Ambjørn, Y.M. Makeenko, J. Nishimura and R.J. Szabo, J. High Energy Phys. 0005 (2000) 023 [hep-th/0004147]; N. Seiberg, J. High Energy Phys. 0009 (2000) 003 [hep-th/0008013].

[4] L. Susskind, [hep-th/0101029]; A.P. Polychronakos, J. High Energy Phys. 0104 (2001) 011 [hep-th/0103013]; S. Hellerman and M. Van Raamsdonk, J. High Energy Phys. 0110 (2001) 039 [hep-th/0103179]; D. Karabali and B. Sakita, [hep-th/0106016]; C.D. Fosco and A. Lopez, [hep-th/0106136]; T.H. Hansson and A. Karlhede, [cond-mat/0109413]; A. Gorsky, I.I. Kogan and C. Korthels-Altes, J. High Energy Phys. 0201 (2002) 002 [hep-th/0111013]; J.L.F. Barbón and A. Paredes, [hep-th/0112185].

[5] Y. Habara, [hep-th/0105002].

[6] V.P. Nair and A.P. Polychronakos, Phys. Lett. B 505 (2001) 267 [hep-th/0011172]; B. Morariu and A.P. Polychronakos, Nucl. Phys. B 610 (2001) 531 [hep-th/0102157; J. Gamboa, M. Loewe, F. Méndez and J.C. Rojas, [hep-th/0104224]; S. Bellucci, A. Nersessian and C. Sochichiu, Phys. Lett. B 522 (2001) 345 [hep-th/0106138]; P.A. Horváthy, [hep-th/0201007].

[7] I.Ya. Aref'eva, D.M. Belov and A.S. Koshelev, [hep-th/0001215]; F. Ruiz Ruiz, [hep-th/0202011].

[8] S. Minwalla, M. Van Raamsdonk and N. Seiberg, J. High Energy Phys. 0002 (2000) 020 [hep-th/9912072].

[9] E. Langmann and R.J. Szabo, in preparation.

[10] R. Gopakumar, S. Minwalla and A. Strominger, J. High Energy Phys. 0005 (2000) 020 [hep-th/0003160].